Problem A.8

Given the following two matrices:

$$\mathsf{A} = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix},$$

compute: (a) $\mathsf{A}+\mathsf{B}$, (b) $\mathsf{A}\,\mathsf{B}$, (c) $[\mathsf{A},\mathsf{B}]$, (d) $\tilde{\mathsf{A}}$, (e) A^*, (f) $\mathsf{A}^\dagger$, (g) $\det(\mathsf{B})$, and (h) B^{-1}. Check that $\mathsf{B}\,\mathsf{B}^{-1} = \mathbf{1}$. Does A have an inverse?

[TYPOS: This should be $\mathsf{B}\,\mathsf{B}^{-1} = \mathsf{I}$ to be consistent with earlier notation.

Also, footnote 9 on page 470 reads, "I'll use a boldface lower-case letters, sans serif, for row and column matrices." Remove the underlined letter.]

Solution

Calculate the desired quantities.

$$\mathsf{A}+\mathsf{B} = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix} = \begin{pmatrix} -1+2 & 1+0 & i-i \\ 2+0 & 0+1 & 3+0 \\ 2i+i & -2i+3 & 2+2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 3i & 3-2i & 4 \end{pmatrix}$$

$$\begin{aligned} \mathsf{A}\,\mathsf{B} &= \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} (-1)(2)+(1)(0)+(i)(i) & (-1)(0)+(1)(1)+(i)(3) & (-1)(-i)+(1)(0)+(i)(2) \\ (2)(2)+(0)(0)+(3)(i) & (2)(0)+(0)(1)+(3)(3) & (2)(-i)+(0)(0)+(3)(2) \\ (2i)(2)+(-2i)(0)+(2)(i) & (2i)(0)+(-2i)(1)+(2)(3) & (2i)(-i)+(-2i)(0)+(2)(2) \end{pmatrix} \\ &= \begin{pmatrix} -3 & 1+3i & 3i \\ 4+3i & 9 & 6-2i \\ 6i & 6-2i & 6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathsf{B}\,\mathsf{A} &= \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix} \\ &= \begin{pmatrix} (2)(-1)+(0)(2)+(-i)(2i) & (2)(1)+(0)(0)+(-i)(-2i) & (2)(i)+(0)(3)+(-i)(2) \\ (0)(-1)+(1)(2)+(0)(2i) & (0)(1)+(1)(0)+(0)(-2i) & (0)(i)+(1)(3)+(0)(2) \\ (i)(-1)+(3)(2)+(2)(2i) & (i)(1)+(3)(0)+(2)(-2i) & (i)(i)+(3)(3)+(2)(2) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 3 \\ 6+3i & -3i & 12 \end{pmatrix} \end{aligned}$$

Calculate the remaining quantities.

$$[\mathsf{A},\mathsf{B}] = \mathsf{AB}-\mathsf{BA} = \begin{pmatrix} -3 & 1+3i & 3i \\ 4+3i & 9 & 6-2i \\ 6i & 6-2i & 6 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 3 \\ 6+3i & -3i & 12 \end{pmatrix} = \begin{pmatrix} -3 & 1+3i & 3i \\ 2+3i & 9 & 3-2i \\ -6+3i & 6+i & -6 \end{pmatrix}$$

$$\tilde{\mathsf{A}} = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix}^T = \begin{pmatrix} -1 & 2 & 2i \\ 1 & 0 & -2i \\ i & 3 & 2 \end{pmatrix}$$

$$\mathsf{A}^* = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix}^* = \begin{pmatrix} -1 & 1 & -i \\ 2 & 0 & 3 \\ -2i & 2i & 2 \end{pmatrix}$$

$$\mathsf{A}^\dagger = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix}^{*T} = \begin{pmatrix} -1 & 1 & -i \\ 2 & 0 & 3 \\ -2i & 2i & 2 \end{pmatrix}^T = \begin{pmatrix} -1 & 2 & -2i \\ 1 & 0 & 2i \\ -i & 3 & 2 \end{pmatrix}$$

$$\det(\mathsf{B}) = \begin{vmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{vmatrix} = -0\begin{vmatrix} 0 & -i \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & -i \\ i & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 \\ i & 3 \end{vmatrix} = (2)(2)-(i)(-i) = 3$$

Find the inverse of B now.

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -i & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ i & 3 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ i & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & \frac{3}{2} & -\frac{i}{2} & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} & -\frac{i}{2} & -3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{i}{3} & -2 & \frac{2}{3} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -i & \frac{i}{3} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{i}{3} & -2 & \frac{2}{3} \end{array}\right]$$

Therefore, the inverse of B is

$$\mathsf{B}^{-1} = \begin{pmatrix} \frac{2}{3} & -i & \frac{i}{3} \\ 0 & 1 & 0 \\ -\frac{i}{3} & -2 & \frac{2}{3} \end{pmatrix}.$$

Check to see that $\mathsf{B}\,\mathsf{B}^{-1} = \mathsf{I}$.

$$\begin{aligned} \mathsf{B}\,\mathsf{B}^{-1} &= \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -i & \frac{i}{3} \\ 0 & 1 & 0 \\ -\frac{i}{3} & -2 & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} (2)\left(\frac{2}{3}\right) + (0)(0) + (-i)\left(-\frac{i}{3}\right) & (2)(-i) + (0)(1) + (-i)(-2) & (2)\left(\frac{i}{3}\right) + (0)(0) + (-i)\left(\frac{2}{3}\right) \\ (0)\left(\frac{2}{3}\right) + (1)(0) + (0)\left(-\frac{i}{3}\right) & (0)(-i) + (1)(1) + (0)(-2) & (0)\left(\frac{i}{3}\right) + (1)(0) + (0)\left(\frac{2}{3}\right) \\ (i)\left(\frac{2}{3}\right) + (3)(0) + (2)\left(-\frac{i}{3}\right) & (i)(-i) + (3)(1) + (2)(-2) & (i)\left(\frac{i}{3}\right) + (3)(0) + (2)\left(\frac{2}{3}\right) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \mathsf{I} \end{aligned}$$

A does not have an inverse because

$$\begin{aligned} \det(\mathsf{A}) = \begin{vmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{vmatrix} &= -2 \begin{vmatrix} 1 & i \\ -2i & 2 \end{vmatrix} + 0 \begin{vmatrix} -1 & i \\ 2i & 2 \end{vmatrix} - 3 \begin{vmatrix} -1 & 1 \\ 2i & -2i \end{vmatrix} \\ &= -2[(1)(2) - (-2i)(i)] - 3[(-1)(-2i) - (2i)(1)] \\ &= -2(0) - 3(0) \\ &= 0. \end{aligned}$$